Supplementary Material

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Introduction

In this supplementary material, we compare the results in the main supplement and in the main text with another common approach to modeling delay discounting: the attribute-wise models (Dai & Busemeyer, 2014). We will detail one model in particular, the Direct Difference Model (DDM; Dai & Busemeyer 2014), so that we can compare the G^2 surface of that model with what we have seen in the other texts. We will see that, while the attribute-wise model we consider here does have some flatness in its G^2 surface – making for more difficult parameter estimation – it is not as drastic as the flatness in the alternative-wise models (and the hyperbolic, in particular). We begin by first discussing some details that both approaches (attribute vs. alternative) share.

Mathematical Details of the Models

Both models fall under the rubric of a *Thurstone Case V* model (Dai & Busemeyer, 2014; Thurstone, 1927), wherein the difference in utility between two options is treated as conceptually similar to the perception of a noisy stimulus. That is, in perceiving noisy stimuli, the subjective value of the stimulus (the internal representation) will not be the same every time the stimulus is perceived, and will hence be judged differently on every trial (Thurstone, 1927). An analogous situation is assumed to occur in ITC; the choice is a stochastic one that may not be repeated every time that same option is shown to the participant (Dai & Busemeyer, 2014; Rieskamp et al., 2006; Rieskamp, 2008). The most common Thurstone models assume that, when comparing two stimuli, the representation of each is drawn from a normal distribution, with (μ_1, σ_1) and (μ_2, σ_2) being the mean and standard deviation of stimulus 1 and 2, respectively (examples of which will be demonstrated in detail soon; Thurstone, 1927). The Case V is when $\sigma_1 = \sigma_2$.

The Stochastic Hyperbolic Model (SHM), assumes that the difference in utility between the larger-later (LL) option and the smaller-sooner (SS) option is a normal random variable with mean given by the hyperbolic difference between the options and a standard deviation parameter. The standard deviation is interpreted as a choice variability parameter, since a large standard deviation implies the random utility difference could be quite far from the expected difference (implying large choice variability), whereas a relatively small standard deviation implies most of the random utility differences will stay close to the expectation. In other words, small σ's indicate high precision in one's choice mechanism. Our mean difference in utility is given by:

$$d := \frac{y^{\rho}}{1+ks} - \frac{x^{\rho}}{1+kt}, \tag{1}$$

we then, by the Thurstone model, have that:[1]

$$P(LL \succsim SS) = \Phi(d/\sigma), \tag{2}$$

where $\sigma > 0$ is the choice variability parameter, and Φ is the standard cumulative normal distribution, $N(0,1)$. The choice variability parameter gives additional information about the choice mechanisms of each subject. That is, placing the hyperbolic model within the context of a general Thurstone Case V model yields probabilities of choice options; these probabilities are governed to a large extent by the parameter values in the model. Taking $\rho = 1$ and $\sigma \to +\infty$ will yield the deterministic hyperbolic model, so that this Thurstone Case V version is one possible generalization of the classical model.

The SHM is classified as an alternative-wise choice mechanism since the mean difference "calculates" the overall utility of an option using the amount and time parameters in the choice option (Dai & Busemeyer, 2014). The alternative-wise assumption is that people evaluate options by arriving at a composite value for each option and then comparing options by their composite scores (Dai & Busemeyer, 2014). The Direct Difference Model (DDM; Dai & Busemeyer, 2014), on the other hand, is one attribute-wise model, wherein options are evaluated based on each attribute, and the final choice is made by summing these attribute scores. The main difference between these two types is that the alternative-wise models come to a "score" by "mixing" the amount and time attributes into one score first, and then summing. The attribute-wise models, instead, evaluate each attribute separately (no "mixing"), and then come to a composite score. The difference, then, is in the order of evaluation. The DDM is also based on a Thurstone Case V model with choice variability

[1] The symbol $\succsim$ denotes the same relationship as in the main supplement. Recall that we represented the LL option as ($y, s delay) and the SS option as ($x, t delay).

similar to the SHM; the crucial difference, however, is the mean difference.

The DDM model comes from a tradition of sequential sampling models of decision making, principally from Decision Field Theory (DFT; Busemeyer & Townsend, 1993). The idea is that preference is a diffusion process where attributes are sampled randomly to accumulate evidence towards one option or the other (Busemeyer & Townsend, 1993). In a similar vein, the DDM model assumes that the attributes of delay and amount are sampled according to the random variable:

$$X := \begin{cases} y^{\rho} - x^{\rho}, \quad \text{with probability } w \\ -(s^{\lambda} - t^{\lambda}) = (t^{\lambda} - s^{\lambda}), \quad \text{with probability } 1 - w. \end{cases} \tag{3}$$

Thus, the model assumes that the advantage (in higher amount) for the LL option is sampled with probability w, whereas the advantage for the SS option is sampled with the complement probability (Dai & Busemeyer, 2014). This gives us an expected difference of:

$$d_{DDM} := E[X] = w(y^{\rho} - x^{\rho}) - (1 - w)(s^{\lambda} - t^{\lambda}), \tag{4}$$

and a standard deviation given by:

$$\sigma^2_{DDM} = E[X^2] - (E[X])^2 = w(y^{\rho} - x^{\rho})^2 + (1 - w)(s^{\lambda} - t^{\lambda})^2 - d^2_{DDM}. \tag{5}$$

This calculation of the standard deviation means that the choice variability parameter is actually a continuous function of the attention weight – a feature not found in the standard probabilistic alternative-wise models. Finally, using these values, we obtain our probability of an LL choice over an SS choice: $P(LL \succsim SS) = \Phi(d_{DDM}/\sigma_{DDM})$. Note here that the expected difference is calculated as a difference in the advantage in amount versus the disadvantage in time for the LL option (hence the attribute-wise nature of the model).[2]

As before, the parameter recovery characteristics of the DDM can be made apparent from the mathematical consequences of the model and its mapping of option values to probabilities.

[2]Here again we assume a non-identity utility function; however, we do so on *both* attributes of delay and amount, with the general assumption that these subjective rescalings are different. (I.e., it is not necessarily the case that $\rho = \lambda$.)

To illustrate the differences between the models, consider the case of an individual with no regard for the time delay, i.e. a "never-discounter." In the SHM, this would mean that $k = 0$, and in the DDM, this participant would focus all his or her attention on the money amount, so that $w = 1$. The intertemporal choice would then reduce to a choice between two monetary amounts, one smaller than the other. We would then expect that the participant would choose the larger option with probability 1.

We first treat this problem using the SHM. Our example suggests we should examine what predictions the SHM model produces as we let $k \to 0$. Indeed, since the standard cumulative normal is a continuous function (as discussed in the main supplement), we let k approach zero in the argument and observe what probability emerges. Specifically:

$$\lim_{k\to 0} \Phi\left(\frac{H(y,s;k) - H(x,t;k)}{\sigma}\right) = \lim_{k\to 0} \Phi\left(\frac{y^{\rho}}{\sigma(1+ks)} - \frac{x^{\rho}}{\sigma(1+kt)}\right) \to \Phi\left(\frac{y^{\rho} - x^{\rho}}{\sigma}\right). \qquad (6)$$

where $H(x,t;k)$ symbolizes the hyperbolic model. Here, the limiting probability is dependent on the utilities of the amount. As $x^{\rho} \to y^{\rho}$, the limiting probability will become $1/2 = \Phi(0)$, if Φ is the standard cumulative normal (or the logistic function used in the main text). However, if the amounts are not close together in value, then the SHM model predicts that there is still a significantly non-zero probability that the never-discounter will choose the smaller reward. It would appear, then, that the SHM does not coincide with our intuition about comparisons between immediate amounts.

In contrast to this, the DDM predicts that the never-discounter will choose the larger option with probability 1. To see this, note that, by Equation (4), as $w \to 1$, so that all attention is put on amount, $d \to (y^{\rho} - x^{\rho})$, which implies that as $w \to 1$, $\sigma^2 \to 0$ (and hence $\sigma \to 0$), by Equation (5). This gives us the equivalence:

$$\lim_{w\to 1} \Phi(d_{DDM}/\sigma_{DDM}) = \lim_{\sigma_{DDM}\to 0} \Phi(d_{DDM}/\sigma_{DDM}), \qquad (7)$$

where the latter equation goes to 0 if $d_{DDM} < 0$ (so that $d_{DDM}/\sigma_{DDM} \to -\infty$ as $\sigma_{DDM} \to 0$) and goes to 1 if $d_{DDM} > 0$ ($d_{DDM}/\sigma_{DDM} \to +\infty$). Thus, if the LL option is less favorable ($d_{DDM} < 0$), it will never be chosen in the limit, while a favorable LL would always be chosen in the limit. We see, then, that the DDM model coincides with our intuitions about immediate options. Note that this difference in outcome came about by the interdependence of σ and w in the DDM (the limiting behaviors go together, not separately; we cannot take the limit of w without also taking the limit of σ).

Finally, recall from the main supplement that we showed how the SHM model "converges" to a random chance model: namely, that this would happen if $k \to +\infty$. An analogous result in the DDM model would be letting $w \to 0$, since this would mean that no attention is put on amount, or that all attention was on the time delay. By Equation (4), this makes $d_{DDM} \to -(s^\lambda - t^\lambda) < 0$, and by Equation (5), $\sigma \to 0$ still. Thus, since $d_{DDM} < 0$ in this case, $d_{DDM}/\sigma \to -\infty \implies \Phi(d_{DDM}/\sigma) \to 0$, so that again, if all attention is put on the time difference, we have $P(LL \succsim SS) \to 0$. This is as expected, since focusing completely on the time aspect of the options makes LL unfavorable in comparison to SS.

Parameter Recovery

In this section we present the parameter recovery statistics for both the SHM and the DDM. Specifically, we generate some data based on parameter estimates provided by (Dai et al., 2016), and we see if a maximum likelihood estimation algorithm can recover these statistics. We use maximum likelihood since it will show fairly clearly how "off the mark" the estimation will be.

In order to have some relatively standard experimental parameters, we applied the DDM model, with parameter estimates given in (Dai et al., 2016), to simulated data based on the Kirby et al. (1999) inventory (Kirby's 27 questions), the same standard used to generate the G^2 surface for the SHM. Thus, these results are most comparable to the results in the main text involving participants who went through the least amount of trials. The general approach is to simulate data with each stochastic model and re-fit the simulated data with the same model. If this re-fitting process yields parameter estimates that are similar to those we used to generate the data, then we have evidence that the parameter estimates are stable (and hence, meaningful). The details of this simulation test go as follows: (1) using the trial details from Kirby et al. (1999), pick some parameters that will simulate responses to the questionnaire (the generating parameters); (2) re-fit the model to the simulated data by repeatedly performing a maximum likelihood estimation until a suitable maximum is achieved; and (3) compare the estimated parameters to the generating parameters. We repeatedly fit the model so that we minimize the chances that we get stuck in a persistent local minimum. This is done by having random initial values for the minimization algorithm (see the MATLAB code that comes with this paper for more details).

We present the results of these tests in Tables S1 and S2 and Figure S1. Each table presents the estimation statistics for each parameter in the models. These are listed as the following columns: the generating value of that parameter, the estimated value after 500 iterations in a minimization program, and the discrepancy True Value - Estimated Value. Table S1 shows these values for the SHM, and Table S2 shows the DDM results. The figure shows the "G^2 surface" for the DDM, as before with the SHM. Recall that the G^2 measure transforms the likelihood estimate into a a log-likelihood in such a way that maximizing the likelihood is equivalent to minimizing the G^2 measure (Busemeyer & Diederich, 2010). In this way, a lower G^2 indicates a better fit to the data. With the simulated data fixed, the surface demonstrates the change in G^2 values across different combinations of parameter values. The leftmost panel shows the G^2 surface, the middle panel shows G^2 changing as a function of w (attention-weight on amount) with the true ρ (utility on amount and time delay) held fix, and the rightmost panel is analogous to the middle, but with w fixed and ρ varying. (Note that in the MATLAB code, what we call ρ here is coded as "Alpha," in line with the parameter names given in Dai & Busemeyer (2014).)

The parameters that we chose for the SHM come from the median estimates for the No-Working Memory load (essentially the "control" condition) as reported in Dai et al. (2016). In this way, the surface yields important insights into real data analysis. The parameters we chose for the DDM are simple parameter estimates for the model. The simplicity is imposing the constraint that $\rho = \lambda$, or, that the utility parameter in the model is equal to the time scaling parameter. While this is not the most general case, we set these parameters equal to effectively reduce the number of parameters in the model so we can obtain a 3D surface of the G^2. (See the relevant MATLAB code for more details.)

First, note from Table S1 that the parameter estimate for k in the SHM could be interpreted as relatively stable. However, a closer look reveals that the discrepancy is $|-0.0013|/(0.0055) \approx .24$, indicating that the estimated value is about 24% greater than the true value. Of course, this problem will be exacerbated for smaller values of k, which are actually very common (Dai et al., 2016; Kirby et al., 1999). The estimation is worse for the σ parameter, where the estimated value is $2.41/8.07 \approx 30\%$ less than the true value. Thus, the fit overestimates k and underestimates σ. Regardless of the direction, these results seem to suggest that the SHM presents some logistical issues if one wants to derive meaningful predictions from these parameter estimates.

SHM	Gen.	Est.	Gen - Est
k	0.0055	0.0068	-0.0013
σ	8.07	5.66	2.41

Table S1
The Maximum Likelihood (ML) parameter estimation statistics for the Stochastic Hyperbolic Model (SHM). The table documents the estimation statistics for the impulsivity parameter k and the choice variability parameter σ. The first column "Gen." is the generating value for the given parameter; these values are taken from Dai et al. (2016), p. 1203. The second column "Est." yields the estimated parameter from the ML method, which is the minimum obtained after 500 repetitions. The third column "Gen. - Est." is the difference between the first and second columns.

A similar story emerges for the DDM model, though such remarks should come with caution because we are analyzing a simplification of the model. The results in Table S2 give us that w is estimated at an about 13% greater value than the true value; that ρ is estimated at an about 46% lesser value; and that λ is estimated at an about 54% lesser value. Thus, w is overestimated, while ρ and λ are (severely) underestimated. This latter result is interesting, especially with our imposed constraint of setting $\rho = \lambda$. While these results are certainly not definitive for the general DDM model, the results suggest that some caution should still be taken when interpreting the parameter estimates.

We proposed the rapid convergence to a random choice model explains why the hyperbolic model, and the alternative-wise models in particular, has trouble recovering parameters. The DDM model fitting does not suffer this same issue, since increasing impulsivity (i.e. decreasing w to 0) does not have the same effect on the probabilities, by the limit results in the previous section. We can see the differences clearly when comparing the surface plots of Figures S1 (the SHM surface in

DDM	Gen.	Est.	Gen - Est
w	0.75	0.85	-0.10
ρ	1	0.54	0.46
λ	1	0.46	0.54

Table S2

The table shows the maximum likelihood estimated parameters for the Direct Difference Model (DDM). The column headers are the same as those in Table 1. Note that these parameter estimates are obtained from the minimum of 500 repetitions of the gradient descent algorithm, as in Table 1. The DDM appears to overestimate w and underestimate ρ *and* λ*.*

the main supplement) and S1 (the DDM surface plotted in this supplement). It is important to note that adding more questions to the Kirby inventory will not ameliorate this issue, as adding more questions will merely increase the spike at $k = 0, \sigma = 0$ for the SHM, and will certainly do nothing to halt the convergence to a random model. This may explain why non-hierarchical method did not outperform the hierarchical approach in the main text, even with more trials, as the hierarchical approach was better at avoiding this decrease in likelihood [increase in G^2] that is inevitable, no matter how many finite questions we have on the test. The problem is a bit trickier for the DDM, asit is not clear if simply constraining the values will help with this problem. Future research could investigate how one might solve this problem in general attribute-wise models.

To summarize, we presented some troubling parameter estimation results concerning both the SHM (the stochastic generalization of the hyperbolic model) and the DDM (an attribute-wise comparison model). Indeed, the estimated parameters can deviate quite strongly from the true generating parameters, which is to be of great concern to practitioners of these models. These mathematical results highlight the need to constrain the region of interest when searching for the best-fitting parameters – whether from a maximum likelihood or Bayesian approach.

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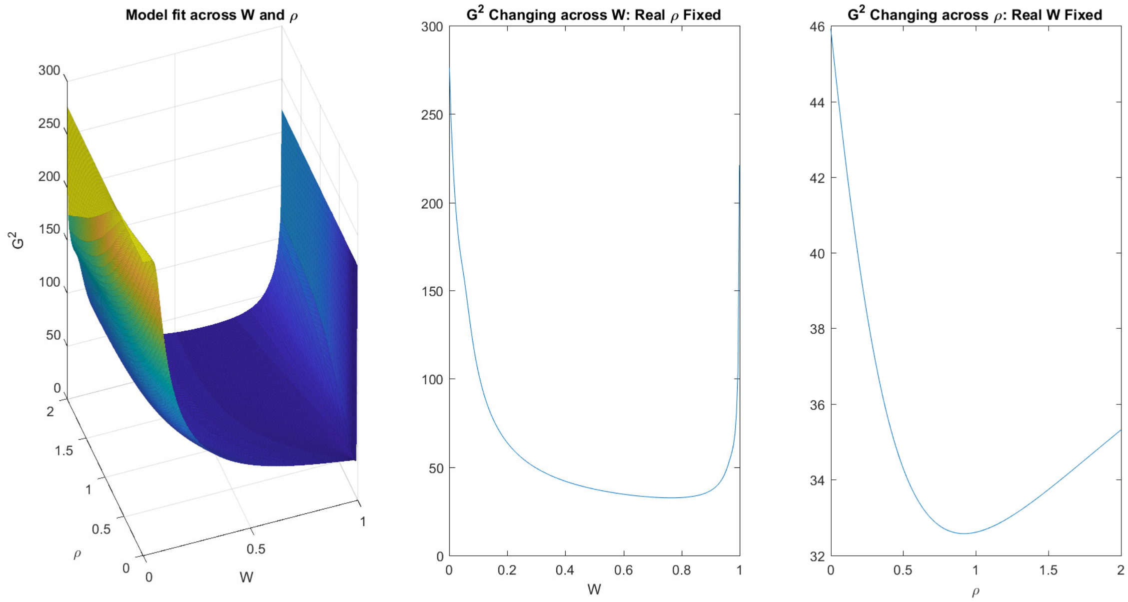

Figure S1. The G^2 goodness-of-fit surface for the Direct Difference Model (DDM). Compare this figure with Figure S1 (the SHM surface). The middle panel shows a slice of the surface at the true ρ value. The right-most panel shows a slice of the surface at the true *w* value. The flatness of the surface near the true parameter values leads to trouble in their estimation.

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